Journal of Engineering Mathematics **33:** 333–352, 1998.

Diffraction of acoustic waves by a semi-infinite cylindrical impedance pipe of certain wall thickness

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Received: 9 December 1996; accepted in revised form: 2 March 1998

Abstract. An asymptotic high-frequency solution is presented for the problem of diffraction of acoustic waves emanating from a ring source by a semi-infinite cylindrical pipe of certain wall thickness having different internal, external and end surface impedances. Through the application of the Fourier-transform technique in conjunction with the Mode Matching method, the diffraction problem is described by a modified Wiener–Hopf equation of the second kind and then solved approximately. Various numerical results illustrating the effects of the parameters of the problem on the diffraction phenomenon are presented.

Keywords: acoustics, pipe, diffraction, Wiener–Hopf technique, high-frequency asymptotics

1. Introduction

The problem of scattering of acoustic waves by semi-infinite hollow cylinders has been extensively studied in the literature because of their use in modelling many actual engineering problems of practical importance, such as the design of a loudspeaker and noise reduction in architectural and experimental aerodynamics, in road transportation, in modern aircraft jet and turbofan engines, etc.

The method which is standard for problem of this type is the *Wiener–Hopf technique* [1]. Levine and Schwinger [2] and Weinstein [3] were the first to apply this method to the study of sound radiation from a semi-infinite rigid pipe. By use of a Green's function, they obtained an integral equation for the velocity potential (*integral-equation technique*) and then solved it exactly by applying the Wiener–Hopf technique, which is originally based on taking first the Fourier transform of the integral equation to derive a functional equation in the spectral domain and then solving it via the factorization procedure. Later, Rawlins [4] obtained the exact solution for the problem of radiation of sound waves from a semi-infinite thin rigid pipe with an internal acoustically absorbent lining for noise reduction studies in acoustic waveguides. The lining is modelled by the acoustic impedance boundary condition. However, modelling a pipe as infinitely thin is unrealistic for most problems and in practice each pipe will have a certain wall thickness which will affect the diffraction phenomenon. The first studies taking this fact into consideration were done by Matsui [5] and Ando [6] for the problem of diffraction of sound waves by an actual microphone system in the case where the semi-infinite cylindrical tube is assumed to be perfectly rigid. According to the method adopted in these works, the diffracted field is expanded into a Dini series (with unknown coefficients) in the waveguide region, while a Fourier integral representation is used elsewhere. The boundary-value problem is first reduced to a modified Wiener–Hopf equation of the second kind and then, after the application of the factorization and decomposition procedure, to the solution of an infinite number of linear algebraic equations involving an infinite number of unknown constants which can always be solved approximately.

In this work an asymptotic high-frequency solution is presented for the problem of diffraction of sound waves emanating from a ring source by a semi-infinite cylindrical pipe of certain wall thickness having different internal, external and end surface impedances. This geometry can be regarded as a more actual model of an acoustic waveguide for use in noise reduction or loudspeaker design studies. The ring source provides the total field to have angular symmetry which makes the problem simpler than the asymmetric case. Through the application of the Fourier-transform technique, the related boundary-value problem can generally be reduced to a pair of simultaneous modified Wiener–Hopf equations of the second kind. It is well known that the crucial step in solving such a matrix Wiener–Hopf equation is the factorization of the kernel matrix as the product of two nonsingular matrices whose entries are regular and of algebraic growth in certain overlapping halves of the complex plane [7]. Except for a very restricted class of matrices, no general method exists to accomplish the Wiener–Hopf factorization of an arbitrary matrix; including the one related to the present problem. In this work, an alternative formulation based on the Mode-Matching method in conjuntion with the Fourier-transform technique will be adopted. This mixed formulation will yield a single (scalar) Wiener–Hopf equation involving an infinite set of unknown constants satisfying an infinite system of linear algebraic equations which are solved approximately. Some computational results illustrating the effects of the admittances and the wall thickness on the diffraction phenomenon are presented.

In the analysis that follows the time dependence is assumed to be $e^{-i\omega t}$ and suppressed throughout this work, where ω is the angular frequency.

2. Formulation of the problem

We consider the scalar wave diffraction by a cylindrical impedance pipe of certain wall thickness occupying the space $a_2 \leqslant \rho \leqslant a_1, z \leqslant 0$ illuminated by a ring source located at $\rho = b > a_1, z = c > 0$ (see Figure 1). Here a_2 and a_1 are the inner and outer radii of the pipe, respectively. We assume that the external surface $\rho = a_1, z < 0$, the internal surface $\rho = a_2, z < 0$ and the surface $a_2 < \rho < a_1, z = 0$ of the cylinder are treated with infinitely thin materials which are characterized by the constant and purely imaginary acoustic surface impedances Z_1, Z_2 and Z_3, respectively.

For analysis purposes, it is convenient to express the total field as follows:

$$u(\rho, z) = \begin{cases} u_1(\rho, z), & \rho > b \\ u_2(\rho, z), & a_1 < \rho < b \\ u_3(\rho, z), & \rho < a_1, z > 0 \\ u_4(\rho, z), & \rho < a_2, z < 0, \end{cases} \tag{1}$$

where u_1, u_2, u_3 and u_4, which satisfy the Helmholtz equation, are to be determined with the aid of the following boundary and continuity conditions:

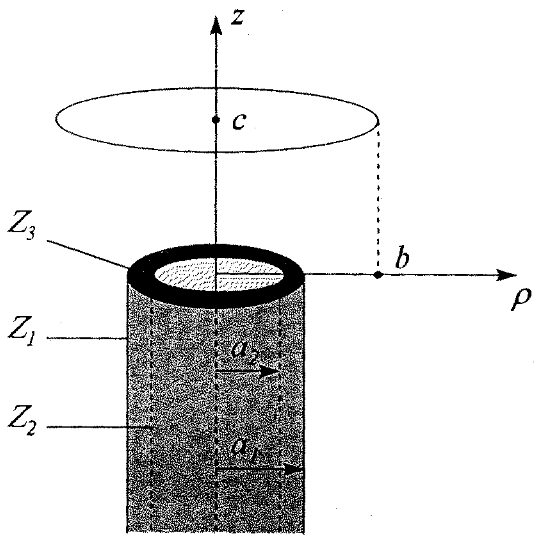

Figure 1. The Geometry of the Diffraction Problem.

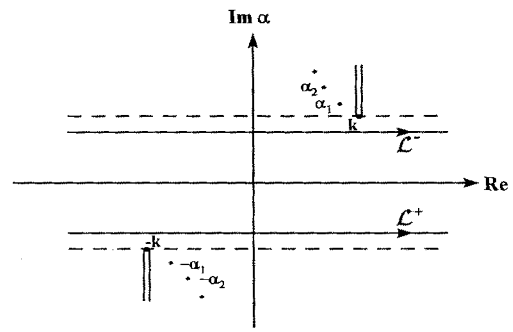

Figure 2. The Complex α-plane.

$$u_1(b,z) = u_2(b,z), \quad z \in (-\infty,\infty), \tag{2a}$$

$$\frac{\partial}{\partial\rho}u_1(b,z) - \frac{\partial}{\partial\rho}u_2(b,z) = \delta(z-c), \quad z \in (-\infty,\infty), \tag{2b}$$

$$\left(ik\eta_1 + \frac{\partial}{\partial\rho}\right)u_2(a_1,z) = 0, \quad z < 0, \tag{2c}$$

$$\left(ik\eta_2 - \frac{\partial}{\partial\rho}\right)u_4(a_2,z) = 0, \quad z < 0, \tag{2d}$$

$$\left(ik\eta_3 + \frac{\partial}{\partial z}\right)u_3(\rho,0) = 0, \quad a_2 < \rho < a_1, \tag{2e}$$

$$u_3(\rho,0) = u_4(\rho,0), \quad \rho < a_2, \tag{2f}$$

$$\frac{\partial}{\partial z}u_3(\rho,0) = \frac{\partial}{\partial z}u_4(\rho,0), \quad \rho < a_2, \tag{2g}$$

$$u_2(a_1,z) = u_3(a_1,z), \quad z > 0, \tag{2h}$$

$$\frac{\partial}{\partial\rho}u_2(a_1,z) = \frac{\partial}{\partial\rho}u_3(a_1,z), \quad z > 0. \tag{2i}$$

In (2c–e) $\eta_i, i = 1,2,3$ are the specific surface admittances and k is the free-space wave number which is temporarily assumed to have a positive imaginary part. To ensure a unique solution we also have to impose edge conditions on the rim $\rho = a_1, z = 0$

$$u_2 = \text{const}, \quad z \to +0, \tag{3a}$$

$$\frac{\partial}{\partial \rho} u_2 = \mathcal{O}(z^{-1/3}), \quad z \to +0. \tag{3b}$$

In the region $\rho > b, u_1(\rho, z)$ satisfies the Helmholtz equation

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right] u_1(\rho, z) = 0, \tag{4}$$

where $z \in (-\infty, \infty)$. Multiplying (4) by $\mathrm{e}^{i\alpha z}$ with α being the Fourier-transform variable and integrating the resulting equation with respect to z from $-\infty$ to ∞, we obtain

$$\left[\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\right) + (k^2 - \alpha^2)\right] F(\rho, \alpha) = 0, \tag{5a}$$

where

$$F(\rho, \alpha) = \int_{-\infty}^{\infty} u_1(\rho, z)\, \mathrm{e}^{i\alpha z}\, \mathrm{d}z. \tag{5b}$$

The solution of (5a) satisfying the radiation condition for $\rho > b$ reads

$$F(\rho, \alpha) = A(\alpha) H_0^{(1)}(K\rho) \tag{6a}$$

with

$$K(\alpha) = \sqrt{k^2 - \alpha^2} \tag{6b}$$

and

$$H_n^{(1)} = J_n + iY_n \tag{6c}$$

being the Hankel function of the first kind and nth order. $A(\alpha)$ in (6a) is a spectral coefficient to be determined. The square-root function $K(\alpha)$ is defined in the complex α-plane cut as shown in Figure 2 such that $K(0) = k$.

In the region $a_1 < \rho < b, u_2(\rho, z)$ satisfies the Helmholtz equation in the range $z \in (-\infty, \infty)$. In a similar way the solution can be given as

$$G^+(\rho, \alpha) + G^-(\rho, \alpha) = B(\alpha) J_0(K\rho) + C(\alpha) Y_0(K\rho) \tag{7a}$$

with $G^{\pm}(\rho, \alpha)$ being defined by

$$G^{\pm}(\rho, \alpha) = \pm \int_0^{\pm\infty} u_2(\rho, z)\, \mathrm{e}^{i\alpha z}\, \mathrm{d}z. \tag{7b}$$

In (7a), $B(\alpha)$ and $C(\alpha)$ are spectral coefficients to be determined. In accordance with the analytical properties of Fourier integrals, the functions $G^-(\rho, \alpha)$ and $G^+(\rho, \alpha)$ are regular functions of α in the half-planes $\mathfrak{Im}(\alpha) < \mathfrak{Im}(k)$ and $\mathfrak{Im}(\alpha) > \mathfrak{Im}(-k)$, respectively. The spectral coefficients $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are related to each other by the definition of the ring source given in (2a, b), the Fourier transform of which give

$$A(\alpha) H_0^{(1)}(Kb) = B(\alpha) J_0(Kb) + C(\alpha) Y_0(Kb) \tag{8a}$$

$$A(\alpha)H_1^{(1)}(Kb) = B(\alpha)J_1(Kb) + C(\alpha)Y_1(Kb) - \frac{e^{i\alpha c}}{K(\alpha)}. \tag{8b}$$

In the Fourier-transform domain (2c) takes the form

$$ik\eta_1 G^-(a_1,\alpha) + \dot{G}^-(a_1,\alpha) = 0, \tag{9}$$

where the dot specifies the derivative with respect to ρ. Utilizing (9) in (7a), we get

$$B(\alpha)M(\alpha) + C(\alpha)N(\alpha) = W^+(\alpha) \tag{10a}$$

with $W^+(\alpha)$, $M(\alpha)$ and $N(\alpha)$ being defined by

$$W^+(\alpha) = ik\eta_1 G^+(a_1,\alpha) + \dot{G}^+(a_1,\alpha), \tag{10b}$$

$$M(\alpha) = ik\eta_1 J_0(Ka_1) - KJ_1(Ka_1), \tag{10c}$$

$$N(\alpha) = ik\eta_1 Y_0(Ka_1) - KY_1(Ka_1). \tag{10d}$$

On the other hand, the elimination of $A(\alpha)$ between (8a) and (8b) gives

$$C(\alpha) - iB(\alpha) = -\frac{\pi b}{2}H_0^{(1)}(Kb)\,e^{i\alpha c}. \tag{11}$$

From (10a) and (11) $B(\alpha)$ and $C(\alpha)$ can be solved uniquely as

$$B(\alpha)L(\alpha) = W^+(\alpha) + \frac{\pi b}{2}N(\alpha)H_0^{(1)}(Kb)\,e^{i\alpha c}, \tag{12a}$$

$$C(\alpha)L(\alpha) = iW^+(\alpha) - \frac{\pi b}{2}M(\alpha)H_0^{(1)}(Kb)\,e^{i\alpha c} \tag{12b}$$

with $L(\alpha)$ being defined by

$$L(\alpha) = ik\eta_1 H_0^{(1)}(Ka_1) - KH_1^{(1)}(Ka_1). \tag{12c}$$

In the region $\rho < a_1$, $u_3(\rho, z)$ satisfies the Helmholtz equation in (4) in the range $z > 0$. Hence, multiplying the Helmholtz equation by $e^{i\alpha z}$ and integrating the resulting equation with respect to z from 0 to ∞, we obtain

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + K^2(\alpha)\right]H^+(\rho,\alpha) = f(\rho) + \alpha g(\rho) \tag{13a}$$

with

$$f(\rho) = \frac{\partial}{\partial z}u_3(\rho,0), \qquad g(\rho) = -iu_3(\rho,0). \tag{13b, c}$$

$H^+(\rho,\alpha)$, which is defined by

$$H^+(\rho,\alpha) = \int_0^\infty u_3(\rho,z)\,e^{i\alpha z}\,dz, \tag{14}$$

is a function regular in the upper half of the complex α-plane ($\Im m(\alpha) > \Im m(-k)$). The particular solution of (13a) which is bounded at $\rho = 0$ and satisfying the impedance boundary condition on $\rho = a_1$ can be obtained by the Green's function technique. The Green's function related to (13a) satisfies the Helmholtz equation

$$\left[\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right) + K^2(\alpha)\right]\mathcal{G}(\rho, t, \alpha) = 0, \quad \rho \neq t, \quad \rho, t \in (0, a_1) \tag{15a}$$

with the following conditions

$$\mathcal{G}(0, t, \alpha) \quad \text{bounded} \tag{15b}$$

$$\mathcal{G}(t + 0, t, \alpha) = \mathcal{G}(t - 0, t, \alpha) \tag{15c}$$

$$\frac{\partial}{\partial\rho}\mathcal{G}(t + 0, t, \alpha) - \frac{\partial}{\partial\rho}\mathcal{G}(t - 0, t, \alpha) = \frac{1}{t} \tag{15d}$$

$$\left(ik\eta_1 + \frac{\partial}{\partial\rho}\right)\mathcal{G}(a_1, t, \alpha) = 0. \tag{15e}$$

The solution is

$$\mathcal{G}(\rho, t, \alpha) = \frac{1}{M(\alpha)}Q(\rho, t, \alpha) \tag{16a}$$

with

$$Q(\rho, t, \alpha) = \frac{\pi}{2}\begin{cases} J_0(K\rho)[M(\alpha)Y_0(Kt) - N(\alpha)J_0(Kt)], & 0 \leqslant \rho \leqslant t \\ J_0(Kt)[M(\alpha)Y_0(K\rho) - N(\alpha)J_0(K\rho)], & t \leqslant \rho \leqslant a_1. \end{cases} \tag{16b}$$

Note that we have

$$ik\eta_1 Q(a_1, t, \alpha) + \dot{Q}(a_1, t, \alpha) = 0. \tag{16c}$$

The solution of (13a) can now be written as

$$H^+(\rho, \alpha) = \frac{1}{M(\alpha)}\left\{D(\alpha)J_0(K\rho) + \int_0^{a_1}[f(t) + \alpha g(t)]Q(t, \rho, \alpha)t\,\mathrm{d}t\right\}. \tag{17}$$

In (17) $D(\alpha)$ is a spectral coefficient to be determined, while f and g are given by (13b, c). Combining (2h) and (2i), we may write

$$ik\eta_1 u_2(a_1, z) + \frac{\partial}{\partial\rho}u_2(a_1, z) = ik\eta_1 u_3(a_1, z) + \frac{\partial}{\partial\rho}u_3(a_1, z), \quad z > 0. \tag{18a}$$

The Fourier transform of (18a) is

$$ik\eta_1 H^+(a_1, \alpha) + \dot{H}^+(a_1, \alpha) = W^+(\alpha). \tag{18b}$$

Substituting (17) and its derivative with respect to ρ in (18b), $D(\alpha)$ can be solved to give

$$D(\alpha) = W^{+}(\alpha). \tag{19}$$

Inserting now (19) into (17) we get

$$H^{+}(\rho,\alpha) = \frac{1}{M(\alpha)}\left[W^{+}(\alpha)J_0(K\rho) + \int_0^{a_1}[f(t)+\alpha g(t)]Q(t,\rho,\alpha)t\,\mathrm{d}t\right]. \tag{20}$$

The left-hand side of (20) is regular in the upper half of the complex α-plane. This implies that the right-hand side should also be regular in the upper half-plane. However, the regularity of the right-hand side in $\Im m(\alpha) > \Im m(-k)$ is violated by the presence of simple poles occurring at the zeros of $M(\alpha)$, namely at $\alpha = \alpha_m$, $m = 1, 2, \dots$ satisfying

$$i\eta_1 k a_1 J_0(\gamma_m) - \gamma_m J_1(\gamma_m) = 0, \quad \alpha_m = \sqrt{k^2 - \left(\frac{\gamma_m}{a_1}\right)^2}, \quad \Im m(\alpha_m) \geqslant \Im m(k). \tag{21}$$

We can eliminate these poles by imposing that their residues are zero. This gives

$$W^{+}(\alpha_m) = \frac{\pi}{2}N(\alpha_m)\frac{a_1^2}{2}J_0^2(\gamma_m)[1-(\eta_1 k a_1/\gamma_m)^2][f_m + \alpha_m g_m] \tag{22a}$$

or

$$W^{+}(\alpha_m) = \frac{a_1}{2}J_0(\gamma_m)[1-(\eta_1 k a_1/\gamma_m)^2][f_m + \alpha_m g_m] \tag{22b}$$

with

$$\begin{bmatrix} f_m \\ g_m \end{bmatrix} = \frac{2}{a_1^2 J_0^2(\gamma_m)[1-(\eta_1 k a_1/\gamma_m)^2]}\int_0^{a_1}\begin{bmatrix} f(t) \\ g(t)\end{bmatrix} J_0\left(\gamma_m \frac{t}{a_1}\right) t\,\mathrm{d}t. \tag{22c}$$

Consider now the waveguide region $\rho < a_2, z < 0$ where the total field can be expressed in terms of Dini series as follows:

$$u_4(\rho,z) = \sum_{n=1}^{\infty} c_n J_0\left(\xi_n \frac{\rho}{a_2}\right)\mathrm{e}^{-i\beta_n z} \tag{23a}$$

with

$$i\eta_2 k a_2 J_0(\xi_n) + \xi_n J_1(\xi_n) = 0, \quad n = 1, 2, \dots \tag{23b}$$

and

$$\beta_n = \sqrt{k^2 - \left(\frac{\xi_n}{a_2}\right)^2}, \quad \Im m(\beta_n) \geqslant \Im m(k). \tag{23c}$$

From the continuity relations (2f) and (13c) we get

$$u_4(\rho,0) = i g(\rho), \quad \rho < a_2. \tag{24}$$

Using (2e), (2f) and (2g) we may write

$$\left(ik\eta_3+\frac{\partial}{\partial z}\right)u_3(\rho,0)=\begin{cases}\left(ik\eta_3+\frac{\partial}{\partial z}\right)u_4(\rho,0), & \rho<a_2\\ 0, & a_2<\rho<a_1.\end{cases} \tag{25}$$

Hence we get

$$f(\rho)-k\eta_3 g(\rho)=\begin{cases}\left(ik\eta_3+\frac{\partial}{\partial z}\right)u_4(\rho,0), & \rho<a_2\\ 0, & a_2<\rho<a_1.\end{cases} \tag{26}$$

Owing to (22c), $f(\rho)$ and $g(\rho)$ can be expanded into Dini series as follows:

$$\begin{bmatrix} f(\rho)\\ g(\rho)\end{bmatrix}=\sum_{m=1}^{\infty}\begin{bmatrix} f_m\\ g_m\end{bmatrix}J_0\left(\gamma_m\frac{\rho}{a_1}\right). \tag{27}$$

Substituting (23a) and (27) in (24) and (26), we obtain

$$i\sum_{m=1}^{\infty}g_m J_0\left(\gamma_m\frac{\rho}{a_1}\right)=\sum_{n=1}^{\infty}c_n J_0\left(\xi_n\frac{\rho}{a_2}\right),\quad \rho<a_2 \tag{28}$$

and

$$\sum_{m=1}^{\infty}(f_m-k\eta_3 g_m)J_0\left(\gamma_m\frac{\rho}{a_1}\right)=\begin{cases}-i\sum_{n=1}^{\infty}c_n(\beta_n-k\eta_3)J_0\left(\xi_n\frac{\rho}{a_2}\right), & \rho<a_2\\ 0, & a_2<\rho<a_1.\end{cases} \tag{29}$$

Let us multiply both sides of (28) by $\rho J_0\left(\xi_\ell\frac{\rho}{a_2}\right)$ and integrate from $\rho=0$ to $\rho=a_2$ to get

$$c_\ell=\frac{2i}{J_0(\xi_\ell)v_\ell}\sum_{m=1}^{\infty}\frac{\Omega_m}{\vartheta_{m\ell}}g_m,\quad \ell=1,2,\ldots \tag{30a}$$

with v_ℓ, Ω_m and $\vartheta_{m\ell}$ being defined by

$$v_\ell=1-(\eta_2 k a_2/\xi_\ell)^2 \tag{30b}$$

$$\Omega_m=i\eta_2 k a_2 J_0(\gamma_m a_2/a_1)+(\gamma_m a_2/a_1)J_1(\gamma_m a_2/a_1) \tag{30c}$$

$$\vartheta_{m\ell}=(\gamma_m a_2/a_1)^2-\xi_\ell^2. \tag{30d}$$

Similarly, the multiplication of both sides of (29) by $\rho J_0\left(\gamma_\ell\frac{\rho}{a_1}\right)$ and its integration from $\rho=0$ to $\rho=a_1$ yields

$$f_\ell-k\eta_3 g_\ell=-2i\left(\frac{a_2}{a_1}\right)^2\frac{\Omega_\ell}{v_\ell J_0^2(\gamma_\ell)}\sum_{n=1}^{\infty}\frac{(\beta_n-k\eta_3)J_0(\xi_n)}{\vartheta_{\ell n}}c_n,\quad \ell=1,2,\ldots \tag{31a}$$

with ν_ℓ being defined by

$$\nu_\ell = 1 - (\eta_1 k a_1/\gamma_\ell)^2. \tag{31b}$$

Consider the continuity relation (2h) which reads, in the Fourier-transform domain

$$H^+(a_1, \alpha) = G^+(a_1, \alpha). \tag{32}$$

Taking into account (7a), (12a,b) and (17), we obtain

$$-\frac{a_1}{2} G^-(a_1, \alpha) + \frac{W^+(\alpha)}{V(\alpha)} = -\frac{b}{2}\frac{H_0^{(1)}(Kb)}{L(\alpha)}\,\mathrm{e}^{i\alpha c} - \frac{1}{2M(\alpha)}\int_0^{a_1}[f(t) + \alpha g(t)]J_0(Kt)t\,\mathrm{d}t \tag{33a}$$

with

$$V(\alpha) = \pi i M(\alpha) L(\alpha). \tag{33b}$$

Substituting (27) in (33a) and evaluating the resulting integral, we obtain the following Modified Wiener–Hopf Equation (MWHE) of the second kind valid in the strip $\mathfrak{Im}(-k) < \mathfrak{Im}(\alpha) < \mathfrak{Im}(k)$

$$-\frac{a_1}{2} G^-(a_1, \alpha) + \frac{W^+(\alpha)}{V(\alpha)} = -\frac{b}{2}\frac{H_0^{(1)}(Kb)}{L(\alpha)}\,\mathrm{e}^{i\alpha c} + \frac{a_1}{2}\sum_{m=1}^{\infty}\frac{J_0(\gamma_m)}{(\alpha_m^2 - \alpha^2)}[f_m + \alpha g_m]. \tag{34}$$

3. Approximate solution of the MWHE

The first step in solving the MWHE is to factorize the kernel function $V(\alpha)$ given in (33b) as

$$V(\alpha) = V^+(\alpha) V^-(\alpha). \tag{35}$$

Here $V^+(\alpha)$ and $V^-(\alpha)$ denote certain functions which are regular and free of zeros in the half-planes $\mathfrak{Im}(\alpha) > \mathfrak{Im}(-k)$ and $\mathfrak{Im}(\alpha) < \mathfrak{Im}(k)$, respectively.

The function $M(\alpha)$ is an entire function with zeros $\alpha = \pm\alpha_m$, $m = 1, 2, \ldots$ given in (21) having the asymptotic behaviour [8, Sections 15.25, 18.33]

$$\alpha_m = \frac{i\gamma_m}{a_1} + O\left(\frac{1}{m}\right) = \frac{(m - 3/4)\pi i}{a_1} + O\left(\frac{1}{m}\right), \quad m \to \infty \tag{36}$$

As a consequence we have the infinite product representation

$$M(\alpha) = M(0)\prod_{m=1}^{\infty}\left(1 - \frac{\alpha^2}{\alpha_m^2}\right) \tag{37}$$

obtained by use of Hadamard's factorization theorem [9, Section 8.24]. The factorization of $M(\alpha)$ is now obvious with the result

$$M^+(\alpha) = [M(0)]^{1/2} \exp\left\{i\frac{\alpha a_1}{\pi}\left[1 - \mathcal{C} + \log\left(\frac{2\pi}{ka_1}\right) + i\frac{\pi}{2}\right]\right\} \times \prod_{m=1}^{\infty}\left(1 + \frac{\alpha}{\alpha_m}\right)\exp\left(\frac{i\alpha a_1}{m\pi}\right) \tag{38a}$$

in which the exponential functions have been deliberately chosen so as to make the infinite product convergent and to produce a simple asymptotic behaviour of $M^+(\alpha)$ as $|\alpha| \to \infty$. In (38a) $\mathcal{C}$ is the Euler's constant given by $\mathcal{C} = 0{\cdot}57721566\ldots$ The asymptotics of $M^+(\alpha)$ is readily found by the procedure from [10, Sections 1–4(III)]

$$M^+(\alpha) \sim \text{Constant} \cdot \alpha^{1/4} \exp\left\{\frac{i\alpha a_1}{\pi}\log\left(\frac{2\alpha}{k}\right)\right\} \quad \text{as} \quad |\alpha| \to \infty \tag{38b}$$

in the upper half-plane.

Application of the factorization procedure from [10, Sections 3–10] to $L(\alpha)$ yields

$$L^+(\alpha) = [L(0)]^{1/2} \exp\left\{-\frac{ika_1}{2} + \frac{a_1 K(\alpha)}{\pi}\log\left(\frac{\alpha + iK(\alpha)}{k}\right) + q(\alpha)\right\} \tag{39a}$$

with the asymptotic behaviour

$$L^+(\alpha) \sim \alpha^{1/4} \exp\left\{-\frac{i\alpha a_1}{\pi}\log\left(\frac{2\alpha}{k}\right)\right\} \quad \text{as} \quad |\alpha| \to \infty \tag{39b}$$

in the upper half-plane. Here, $q(\alpha)$ is given by

$$q(\alpha) = \frac{1}{\pi}\int_0^{\infty}\left[1 - \frac{2}{\pi x}\frac{x^2 - (\eta_1 k a_1)^2}{[(i\eta_1 k a_1 J_0(x) - xJ_1(x))^2 + (i\eta_1 k a_1 Y_0(x) - xY_1(x))^2]}\right] \times \log\left(1 + \frac{\alpha a_1}{\sqrt{(ka_2)^2 - x^2}}\right)\mathrm{d}x \tag{39c}$$

and it has been assumed that $L(\alpha)$ has no zeros in the complex α-plane. On multiplying the results for $M^+(\alpha)$ and $L^+(\alpha)$, we get

$$\begin{aligned} V^+(\alpha) = {} & [\pi i(i\eta_1 k J_0(ka_1) - kJ_1(ka_1))(i\eta_1 k H_0^{(1)}(ka_1) - kH_1^{(1)}(ka_1))]^{1/2} \\ & \times \exp\left\{i\frac{\alpha a_1}{\pi}\left[1 - \mathcal{C} + \log\left(\frac{2\pi}{ka_1}\right) + i\frac{\pi}{2}\right] - i\frac{ka_1}{2}\right\} \\ & \times \exp\left\{\frac{a_1 K(\alpha)}{\pi}\log\left(\frac{\alpha + iK(\alpha)}{k}\right) + q(\alpha)\right\} \\ & \times \prod_{m=1}^{\infty}\left(1 + \frac{\alpha}{\alpha_m}\right)\exp\left(\frac{i\alpha a_1}{m\pi}\right). \end{aligned} \tag{40a}$$

$$V^-(\alpha) = V^+(-\alpha) \tag{40b}$$

with the asymptotic behaviour

$$V^\pm(\alpha) \sim (\pm\alpha)^{1/2} \quad \text{as} \quad |\alpha| \to \infty \tag{40c}$$

in their respective regions of regularity.

The multiplication of both sides of (34) by $V^-(\alpha)$ and decomposition of the resulting equation into the Wiener–Hopf form leads to

$$\begin{aligned}
&\frac{W^+(\alpha)}{V^+(\alpha)} + \frac{1}{2\pi i}\frac{b}{2}\int_{\mathcal{L}^+} V^-(\tau)\frac{H_0^{(1)}(Kb)}{L(\tau)}\frac{\mathrm{e}^{i\tau c}}{(\tau-\alpha)}\,\mathrm{d}\tau - \frac{a_1}{2}\sum_{m=1}^{\infty}\frac{J_0(\gamma_m)}{2\alpha_m}\frac{V^+(\alpha_m)}{(\alpha+\alpha_m)}[f_m - \alpha_m g_m] \\
&= \frac{a_1}{2}V^-(\alpha)G^-(a_1,\alpha) + \frac{1}{2\pi i}\frac{b}{2}\int_{\mathcal{L}^-} V^-(\tau)\frac{H_0^{(1)}(Kb)}{L(\tau)}\frac{\mathrm{e}^{i\tau c}}{(\tau-\alpha)}\,\mathrm{d}\tau \\
&\quad - \frac{1}{2\pi i}\frac{a_1}{2}\sum_{m=1}^{\infty} J_0(\gamma_m)\int_{\mathcal{L}^-} V^-(\tau)\frac{[f_m+\tau g_m]}{(\alpha_m^2-\tau^2)}\frac{\mathrm{d}\tau}{(\tau-\alpha)}.
\end{aligned} \tag{41}$$

The position of the integration lines $\mathcal{L}^+$ and $\mathcal{L}^-$ are depicted in Figure 2. The left- and right-hand sides of (41) are regular in the lower ($\Im m(\alpha) < \Im m(k)$) and the upper ($\Im m(\alpha) > \Im m(-k)$) halves of the complex α-plane, respectively. Therefore, by the analytical continuation principle, they define an entire function which can be shown to be zero. From (41), equating the left-hand side to zero, we have

$$\frac{W^+(\alpha)}{V^+(\alpha)} = I(\alpha) + \frac{a_1}{2}\sum_{m=1}^{\infty}\frac{J_0(\gamma_m)}{2\alpha_m}\frac{V^+(\alpha_m)}{(\alpha+\alpha_m)}[f_m - \alpha_m g_m] \tag{42a}$$

with $I(\alpha)$ being defined by

$$I(\alpha) = -\frac{1}{2\pi i}\frac{b}{2}\int_{\mathcal{L}^+} V^-(\tau)\frac{H_0^{(1)}(Kb)}{L(\tau)}\frac{\mathrm{e}^{i\tau c}}{(\tau-\alpha)}\,\mathrm{d}\tau. \tag{42b}$$

The integral given by (42b) can be evaluated by means of the steepest-descent method. The substitution $\tau = -k\cos\xi$ in (42b) enables us to write

$$I(\alpha) = \frac{1}{2\pi i}\frac{b}{2}\int_{\Gamma} V^-(-k\cos\xi)\frac{H_0^{(1)}(kb\sin\xi)}{L(-k\cos\xi)}\frac{\mathrm{e}^{-ikc\cos\xi}}{(\alpha+k\cos\xi)}k\sin\xi\,\mathrm{d}\xi, \tag{43}$$

where Γ is the new integration line depicted in Figure 3.

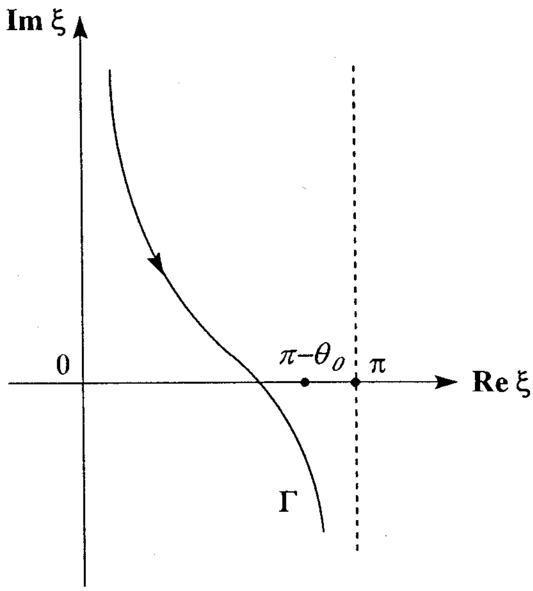

Figure 3. The Complex ξ-plane.

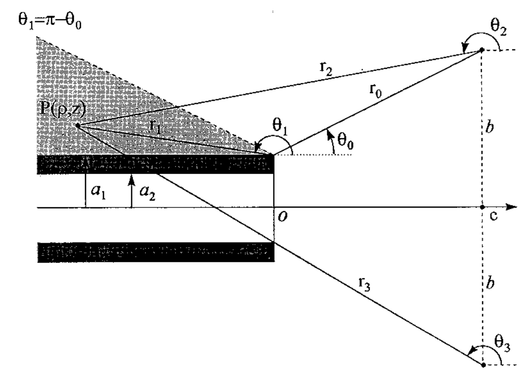

Figure 4. Total Field at a Point $P(\rho, z)$.

Using the asymptotic expansion of Hankel's functions for large arguments and making the following substitutions

$$b - a_1 = r_0 \sin\theta_0, \qquad c = r_0 \cos\theta_0$$

in (43), we get

$$I(\alpha) = -\frac{\sqrt{a_1 b}}{4\pi} \int_\Gamma V^-(-k\cos\xi) \frac{\sin\xi}{\eta_1 + \sin\xi} \frac{\mathrm{e}^{-ikr_0\cos(\xi+\theta_0)}}{(\alpha + k\cos\xi)} \mathrm{d}\xi. \tag{44}$$

During the deformation of the integration line Γ onto the steepest-descent path (SDP) through the saddle point $\xi_s = \pi - \theta_0$, we may cross the pole occurring at $\xi = \arccos(-\alpha/k)$ whose residue contribution is

$$I_{\text{res}}(\alpha) = -\frac{b}{2} V^-(\alpha) \frac{H_0^{(1)}(Kb)}{L(\alpha)} \mathrm{e}^{i\alpha c} \mathcal{H}[\Im m(k\cos\theta_0 - \alpha)], \tag{45a}$$

where $\mathcal{H}$ denotes the unit-step function. The contribution from the SDP, *i.e.*

$$I_d(\alpha) = -\frac{\sqrt{a_1 b}}{4\pi} \int_{\text{SDP}} V^-(-k\cos\xi) \frac{\sin\xi}{\eta_1 + \sin\xi} \frac{\mathrm{e}^{ikr_0\cos(\xi+\theta_0)}}{(\alpha + k\cos\xi)} \mathrm{d}\xi \tag{45b}$$

can be obtained by evaluation of (45b) through the saddle-point formula. Indeed, as $kr_0 \to \infty$, we have

$$I_d(\alpha) \simeq \frac{\mathrm{e}^{i3\pi/4}}{\sqrt{2\pi}} \frac{\sqrt{a_1 b}}{2} \frac{\sin\theta_0}{\eta_1 + \sin\theta_0} \frac{V^-(k\cos\theta_0)}{(\alpha - k\cos\theta_0)} \frac{\mathrm{e}^{ikr_0}}{kr_0}. \tag{45c}$$

So $I(\alpha)$ can be given as

$$I(\alpha) = I_{\text{res}}(\alpha) + I_d(\alpha). \tag{45d}$$

Substituting $\alpha = \alpha_1, \alpha_2, \ldots$ in (42a) and using (22b), we have the following equations for f_r and g_r

$$\frac{J_0(\gamma_r)\nu_r}{V^+(\alpha_r)}[f_r + \alpha_r g_r]$$
$$= \frac{2}{a_1} I(\alpha_r) + \sum_{m=1}^{\infty} \frac{J_0(\gamma_m)}{2\alpha_m} \frac{V^+(\alpha_m)}{(\alpha_r + \alpha_m)}[f_m - \alpha_m g_m], \quad r = 1, 2, \ldots \tag{46}$$

Replacing (30a) in (31a), we may express f_r in terms of g_r as

$$f_r = k\eta_3 g_r + 4\left(\frac{a_2}{a_1}\right)^2 \frac{\Omega_r}{\nu_r J_0^2(\gamma_r)} \sum_{m=1}^{\infty} g_m \Omega_m \sum_{n=1}^{\infty} \frac{\beta_n - k\eta_3}{v_n \vartheta_{mn} \vartheta_{rn}}. \tag{47}$$

Substituting (47) in (46) we get infinitely many equations in infinite number of unknowns which yield the constants g_r, $r = 1, 2, \ldots$ as follows:

$$\left[\frac{J_0(\gamma_r)(\alpha_r + k\eta_3)\nu_r}{2V^+(\alpha_r)} + C_r(\alpha_r)\right] g_r + \sum_{\substack{m=1 \\ m\neq r}}^{\infty} C_m(\alpha_r) g_m = \frac{1}{a_1} I(\alpha_r), \quad r = 1, 2, \ldots \tag{48a}$$

with

$$C_m(\alpha_r) = \frac{J_0(\gamma_m)V^+(\alpha_m)(k\eta_3 - \alpha_m)}{4\alpha_m(\alpha_r + \alpha_m)} + \left(\frac{a_2}{a_1}\right)^2 \Omega_m \sum_{n=1}^{\infty} \frac{\beta_n - k\eta_3}{v_n \vartheta_{mn}}$$
$$\times \left\{ \frac{2\Omega_r}{V^+(\alpha_r) J_0(\gamma_r) \vartheta_{rn}} - \sum_{s=1}^{\infty} \frac{V^+(\alpha_s)\Omega_s}{\alpha_s(\alpha_r + \alpha_s) J_0(\gamma_s) \nu_s \vartheta_{sn}} \right\}. \tag{48b}$$

In the numerical implementations the infinite sums given in (46) and (47) are truncated after the Nth term and the infinite system of equations in (48a,b) is solved approximately.

4. Asymptotic evaluation of the total field

The total field in the region $\rho > b$ can be obtained from the inverse Fourier transform of $F(\rho, \alpha)$. From (6a) we get

$$u_1(\rho, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\alpha) H_0^{(1)}(K\rho)\, \mathrm{e}^{-i\alpha z}\, \mathrm{d}\alpha. \tag{49}$$

Using (8a), (12a,b), (42a) and (45a,c,d), we may write the total field as the sum of the following four terms:

$$u_1(\rho, z) = u^i(\rho, z) + u^r(\rho, z) + u_{d1}(\rho, z) + u_{d2}(\rho, z) \tag{50a}$$

where

$$u^i(\rho, z) = \frac{b}{4i} \int_{-\infty}^{\infty} J_0(Kb) H_0^{(1)}(K\rho)\, \mathrm{e}^{-i\alpha(z-c)}\, \mathrm{d}\alpha, \tag{50b}$$

$$\begin{aligned} u^r(\rho, z) \;=\; & \frac{b}{4i} \int_{-\infty}^{\infty} \frac{M(\alpha)}{L(\alpha)} H_0^{(1)}(Kb)\, \mathrm{e}^{i\alpha c} \{\mathcal{H}[\mathfrak{J}m(k\cos\theta_0 - \alpha)] - 1\} \\ & \times H_0^{(1)}(K\rho)\, \mathrm{e}^{-i\alpha z}\, \mathrm{d}\alpha, \end{aligned} \tag{50c}$$

$$\begin{aligned} u_{d1}(\rho, z) \;=\; & \frac{\mathrm{e}^{i3\pi/4}}{\sqrt{2\pi}} \frac{\sqrt{a_1 b}}{4\pi} \frac{\sin\theta_0}{\eta_1 + \sin\theta_0} V^-(k\cos\theta_0) \\ & \times \frac{\mathrm{e}^{ikr_0}}{kr_0} \int_{-\infty}^{\infty} \frac{V^+(\alpha)}{L(\alpha)} \frac{H_0^{(1)}(K\rho)}{(\alpha - k\cos\theta_0)}\, \mathrm{e}^{-i\alpha z}\, \mathrm{d}\alpha, \end{aligned} \tag{50d}$$

$$\begin{aligned} u_{d2}(\rho, z) \;=\; & \frac{a_1}{8\pi} \sum_{m=1}^{\infty} \frac{J_0(\gamma_m)}{\alpha_m} V^+(\alpha_m)[f_m - \alpha_m g_m] \\ & \times \int_{-\infty}^{\infty} \frac{V^+(\alpha)}{L(\alpha)} \frac{H_0^{(1)}(K\rho)}{(\alpha + \alpha_m)}\, \mathrm{e}^{-i\alpha z}\, \mathrm{d}\alpha. \end{aligned} \tag{50e}$$

Using the asymptotic expansion of Hankel's functions for large arguments, we observe that the asymptotic evaluation of the integrals in (50b,e) through the saddle-point technique yields for the total field

$$u_1(\rho, z) = u^i(r_2, \theta_2; r_3, \theta_3) + u^r(r_1, \theta_1) + u_{d1}(r_1, \theta_1) + u_{d2}(r_1, \theta_1), \tag{51a}$$

where

$$u^i(r_2, \theta_2; r_3, \theta_3) = \frac{\mathrm{e}^{i3\pi/4}}{\sqrt{2\pi}} \frac{\sqrt{kb}}{2} \left[\frac{\mathrm{e}^{ikr_2}}{kr_2} - i\, \frac{\mathrm{e}^{ikr_3}}{kr_3} \right], \tag{51b}$$

$$\begin{aligned} u^r(r_1, \theta_1) \;=\; & \sqrt{2\pi}\, \mathrm{e}^{i3\pi/4} \frac{b}{4} \sqrt{ka_1} \frac{\sin\theta_1}{\eta_1 + \sin\theta_1} H_0^{(1)}(kb\sin\theta_1)\, \mathrm{e}^{-ikc\cos\theta_1} \\ & \times \{\mathcal{H}[\mathfrak{J}m(k(\cos\theta_0 + \cos\theta_1))] - 1\} \frac{\mathrm{e}^{ikr_1}}{kr_1}, \end{aligned} \tag{51c}$$

$$\begin{aligned} u_{d1}(r_1, \theta_1) \;=\; & -\frac{ka_1\sqrt{kb}}{4\pi k^2} \frac{\mathrm{e}^{ikr_0}}{kr_0} \frac{\sin(\theta_0/2)}{\eta_1 + \sin\theta_0} \frac{\sin(\theta_1/2)}{\eta_1 + \sin\theta_1} V^-(k\cos\theta_0) V^-(k\cos\theta_1) \\ & \times \left\{ \sec\left(\frac{\theta_1 - \theta_0}{2}\right) \mathcal{F}\left(2kr_1 \cos^2\left(\frac{\theta_1 - \theta_0}{2}\right)\right) \right. \\ & \left. + \sec\left(\frac{\theta_1 + \theta_0}{2}\right) \mathcal{F}\left(2kr_1 \cos^2\left(\frac{\theta_1 + \theta_0}{2}\right)\right) \right\} \frac{\mathrm{e}^{ikr_1}}{kr_1}, \end{aligned} \tag{51d}$$

$$u_{d2}(r_1,\theta_1) = \sqrt{2\pi}\, e^{-i3\pi/4}\frac{a_1}{8\pi}\sqrt{ka_1}\frac{\sin\theta_1}{\eta_1+\sin\theta_1}V^-(k\cos\theta_1)\frac{e^{ikr_1}}{kr_1} \times \sum_{m=1}^{\infty}\frac{J_0(\gamma_m)}{\alpha_m}\frac{V^+(\alpha_m)}{(\alpha_m-k\cos\theta_1)}[f_m-\alpha_m g_m] \tag{51e}$$

with

$$\mathcal{F}(z) = -2i\sqrt{z}\, e^{-iz}\int_{\sqrt{z}}^{\infty} e^{-it^2}\, dt \tag{51f}$$

begin the well-known Fresnel integral and $r_1, \theta_1, r_2, \theta_2$ and r_3, θ_3 appearing in (51a–e) are the spherical coordinates defined by

$$\rho - a_1 = r_1\sin\theta_1, \qquad z = r_1\cos\theta_1$$

$$\rho - b = r_2\sin\theta_2, \qquad z - c = r_2\cos\theta_2$$

and

$$\rho + b = r_3\sin\theta_3, \qquad z - c = r_3\cos\theta_3$$

as shown in Figure 4.

In the expression of the total field given in (51a) the first term at the right-hand side corresponds to the incident wave emanating from the ring source. The second term is the reflected field from the lateral surface $\rho = a_1, z < 0$ of the cylinder. The reflected field appears only in the region $\pi - \theta_0 < \theta_1 < \pi$ as shown in Figure 4. The third and the fourth terms together account for the total diffracted field. Equation (51f) is the approximate expression of the incident field at the rim $\rho = a_1, z = 0$, as $kr_0 \to \infty$.

5. Some special cases

In this section the results related to some special cases which are of theoretical as well as practical importance will be discussed.

(i) $\eta_1 = \eta_2 = 0, \ \ a_2 < a_1.$

Consider first the special case of a perfectly rigid pipe with a certain wall thickness and an impedance end. For this special case we get

$$\gamma_1 = \xi_1 = 0, \qquad \gamma_m = \xi_m > 0, \qquad m = 2, 3, \ldots \tag{52}$$

Besides replacing $\eta_1 = \eta_2 = 0$ in the formulations at every step, we must make the following modifications: First of all, the terms c_1 and f_1 in (30a) and (31a) should be calculated separately from the other terms c_ℓ and $f_\ell, \ell > 1$. Multiplication of both sides of (28) and (29) by ρ and its integration from $\rho = 0$ to $\rho = a_2$ and $\rho = a_1$, respectively gives

$$c_1 = ig_1 + i2\frac{a_1}{a_2}\sum_{m=2}^{\infty}\frac{J_1(\gamma_m a_2/a_1)}{\gamma_m}g_m \tag{53a}$$

$$f_1 - k\eta_3 g_1 = -ik\left(\frac{a_2}{a_1}\right)^2 (1-\eta_3)c_1. \tag{53b}$$

Replacing (53b) in (53a), we can express f_1 in terms of g_m as

$$f_1 = k\left[\eta_3 + \left(\frac{a_2}{a_1}\right)^2 (1-\eta_3)\right] g_1 + 2k\left(\frac{a_2}{a_1}\right)(1-\eta_3)\sum_{m=2}^{\infty}\frac{J_1(\gamma_m a_2/a_1)}{\gamma_m} g_m. \tag{54}$$

Finally, replacing (54) in (46), we have for $r = 1$

$$\frac{k}{V^+(k)}\left\{\left[1+\left(\frac{V^+(k)}{2k}\right)^2\right] + \left[1-\left(\frac{V^+(k)}{2k}\right)^2\right]\left[\eta_3 + \left(\frac{a_2}{a_1}\right)^2 (1-\eta_3)\right]\right\} g_1$$
$$+\sum_{m=2}^{N} C_m(k) g_m = \frac{2}{a_1} I(k) \tag{55a}$$

with

$$\begin{aligned} C_m(k) \;=\; & \frac{2k}{V^+(k)}\left[1-\left(\frac{V^+(k)}{2k}\right)^2\right]\left(\frac{a_2}{a_1}\right)^2 (1-\eta_3)\frac{J_1(\gamma_m a_2/a_1)}{\gamma_m} \\ & -\frac{J_0(\gamma_m)V^+(\alpha_m)}{2(k+\alpha_m)}\left(1+\frac{k\eta_3}{\alpha_m}\right) \\ & -2\left(\frac{a_2}{a_1}\right)^4 \frac{\gamma_m J_1(\gamma_m a_2/a_1)}{J_0^2(\gamma_m)}\sum_{s=2}^{N}\frac{V^+(\alpha_2)\gamma_s J_1(\gamma_s a_2/a_1)}{\alpha_s(k+\alpha_s)J_0(\gamma_s)}\sum_{n=1}^{N}\frac{\beta_n - k\eta_3}{\vartheta_{mn}\vartheta_{sn}}. \end{aligned} \tag{55b}$$

(ii) $\eta_1 + \eta_2 = 0, \;\; a_2 = a_1.$

For this special case we get

$$\alpha_m = \beta_m, \qquad \gamma_m = \xi_m, \qquad m = 1, 2, \ldots$$

Therefore, after putting $\eta_3 = 0$ and $a_2 = a_1$ at every step, we see that (28) and (29) are identically satisfied for

$$i g_m = c_m \quad \text{and} \quad f_m = -i c_m \beta_m.$$

This gives

$$f_m = \alpha_m g_m$$

implying $u_{d2} = 0$ in (51a). Hence, $D(\theta_0, \theta_1)$ given in (51g) corresponds to the 'diffraction coefficient' related to the rim $\rho = a_1, z = 0$ of the cylinder for this case.

(iii) $a_2 = 0.$

For the special case of a semi-infinite impedance rod, we set

$$c_n \equiv 0, \qquad n = 1, 2, \ldots$$

This yields

$$f_n = k\eta_3 g_n$$

and

$$C_m(\alpha_r) = \frac{J_0(\gamma_m)V^+(\alpha_m)(k\eta_3 - \alpha_m)}{4\alpha_m(\alpha_r + \alpha_m)}$$

in the formulations. When $\eta_1 = \eta_3 = 0$, the solution of the WHE reduces to

$$G^-(a_1, \alpha) = \frac{2W^+(\alpha)}{a_1 K^2(\alpha)R(\alpha)} - \frac{b}{a_1}\frac{H_0^{(1)}(Kb)}{KH_1^{(1)}(Ka_1)}\,\mathrm{e}^{i\alpha c} + \alpha\sum_{m=1}^{\infty}\frac{J_0(\gamma_m)}{(\alpha^2 - \alpha_m^2)}g_m$$

with

$$R(\alpha) = -\pi i J_1(Ka_1)H_1^{(1)}(Ka_1),$$

which agrees with the solution of the WHE derived by D. S. Jones for the problem of diffraction by a semi-infinite rigid rod of circular cross-section [11].

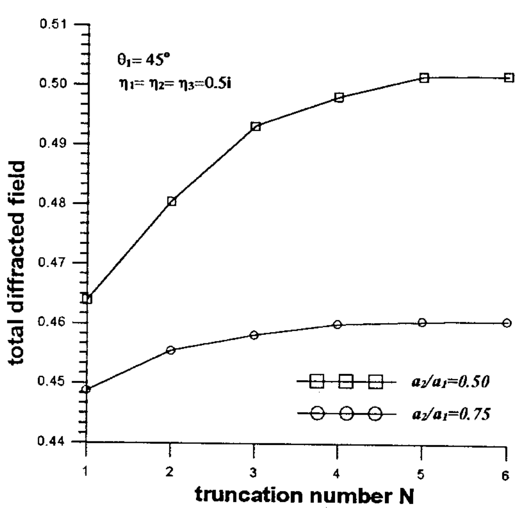

Figure 5. Total diffracted field versus the truncation number N.

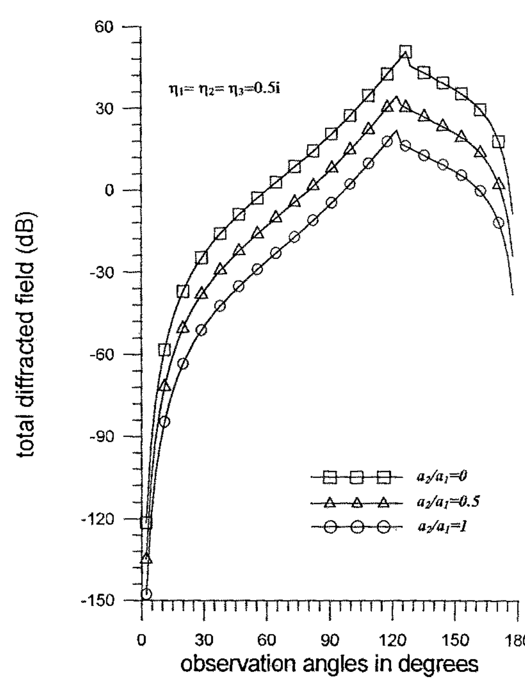

Figure 6. Total diffracted field versus the observation angle θ_1, for different values of a_2/a_1.

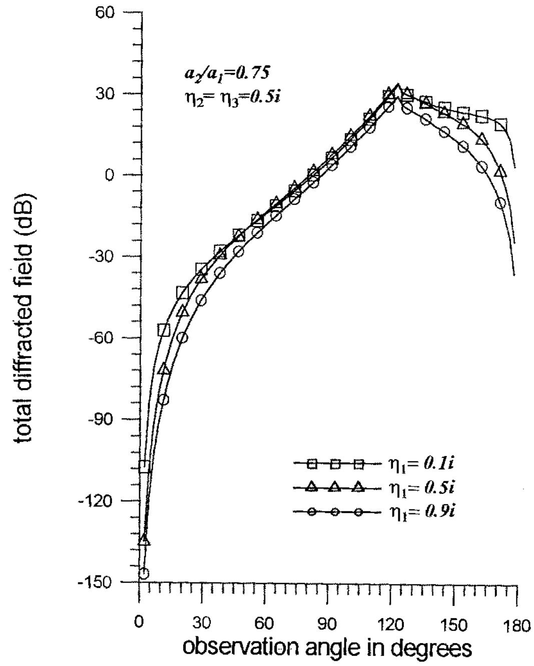

Figure 7. Total diffracted field versus the observation angle θ_1, for different values of η_1.

In order to show the influence of the values of the wall thickness and the surface impedances on the diffraction phenomenon, some numerical results showing the variation of the total diffracted field (20 $\log_{10}|u_d|$) with the observation angle are presented. In all the graphical solutions that follows we use dimensionless coordinates and take $ka_1 = 1, kb = 10, kc = 6$, and $kr_1 = 10$. In Figure 5 we show the variation of the total diffracted field with the truncation number N at a fixed point. It is seen that the total diffracted field becomes insensitive to the truncation number N for $N \geqslant 5, a_2/a_1 = 0{\cdot}50, \eta_1 = \eta_2 = \eta_3 = i0{\cdot}5$. The reason of this rapid convergence is the very small contribution of u_{d2} term (a few percent) to the total diffracted field. This fact is also in agreement with the numerical convergence tests given in the papers of Rawlins [4] and Ando [6]. We require a smaller N for increasing values of a_2/a_1 as expected. For the numerical examples that follow N is chosen with this criterion in mind.

Another interesting result is that the total diffracted field is insensitive to the variations of the surface impedances Z_2 and Z_3, since they appear only in the u_{d2} term. So it can be deduced that the total diffracted field is affected merely by the variations in the wall thickness a_2/a_1 and the external surface impedance Z_1. Figure 6 shows the variation of the total diffracted field with the observation angle, for various values of the wall thickness a_2/a_1. As expected, the total diffracted field increases regularly with increasing value of the wall thickness. This shows the importance of the contribution of the wall thickness of a realistic pipe to the diffraction phenomenon. Figure 7 shows the variation of the total diffracted field with the observation angle, for various values of the exterior surface admittance η_1, which is taken as positive imaginary ($\eta_1 = i\zeta_1, \zeta_1 > 0$). The total diffracted field decreases with the increasing values of ζ_1. Figure 8 shows the variation of the total diffracted field with the observation angle, for various values of the exterior surface admittance η_1, which is taken as negative imaginary ($\eta_1 = -i\zeta_1, \zeta_1 > 0$). The total diffracted field decreases with the increasing values of ζ_1.

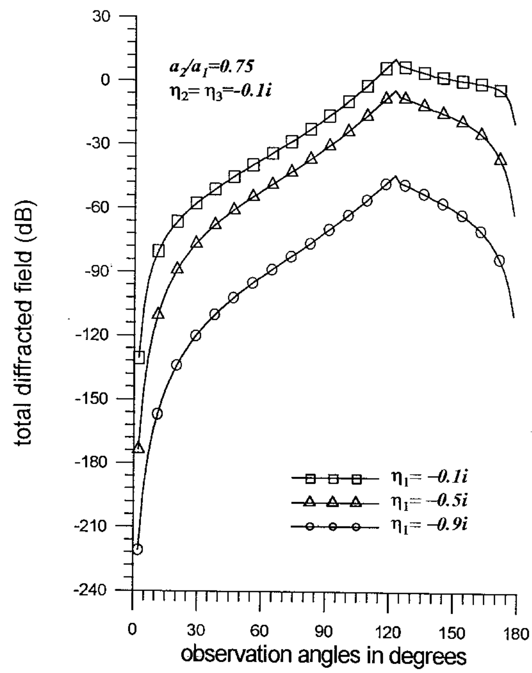

Figure 8. Total diffracted field versus the observation angle θ_1, for different values of η_1.

6. Conclusions

In this work, an asymptotic high-frequency solution is presented for the problem of scalar wave diffraction by a semi-infinite pipe of certain wall thickness having different internal, external and end surface impedances, through application of the Wiener–Hopf technique. As far as we know, no studies have been reported in the literature for the problem of diffraction by cylindrical waveguides satisfying an external impedance boundary condition. Extension of the problem to the case when the lateral surface admittances η_1 and η_2 have positive real parts (absorbent coating), requires that it should be verified beforehand that the functions $J_0(\gamma_m t)$ and $J_0(\xi_n t)$ form *complete* systems on $0 \leqslant t \leqslant 1$, thus permitting expansions in Dini series as in (23a) and (27). In our solution we also present the factorization of the kernel function $V(\alpha)$ which is always required in problems concerning diffraction by semi-infinite or finite-length cylindrical impedance pipes or rods. The problem can possibly by generalized to the finite-length case. In that case, the solution of the modified Wiener–Hopf equation involves branch-cut integrals with unknown integrands which have to be performed numerically, and infinitely many constants satisfying infinite systems of linear algebraic equations.

Acknowledgements

The authors are indebted to Prof. Dr. J. Boersma from Eindhoven University of Technology (The Netherlands) for his constructive comments as a referee which led to the improvement of this work.

This work is partially supported by the Turkish Academy of Sciences.

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